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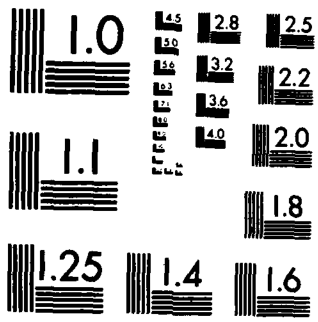
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MRC Technical Summary Report #2648

A NOTE ON EQUALITY IN ANDERSON'S THEOREM

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February 1984

(Received January 16, 1984)

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UNIVERSITY OF WISCONSIN-MADISON
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ABSTRACT

In this note it is shown, by a counterexample, that the necessary and sufficient condition for equality in Anderson's theorem, given by Anderson (1955), is incorrect. A more general condition is given which is shown to be necessary and sufficient. This is applied to the multivariate normal distribution.

AMS (MOS) Subject Classifications: 60E15, 62H99

Key Words: Anderson's theorem, convexity, elliptically contoured distributions, multivariate normal distribution, symmetry about the origin, unimodality

Work Unit Number 4 (Statistics and Probability)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the University of Wisconsin-Milwaukee.

SIGNIFICANCE AND EXPLANATION

Anderson's theorem is a widely used tool to obtain multivariate inequalities. Sometimes it is important to know when these inequalities are strict. This note gives a corrected necessary and sufficient condition for this to be the case.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NOTE ON EQUALITY IN ANDERSON'S THEOREM

Andrew P. Soms

1. INTRODUCTION

Anderson's theorem (1955) is an important and widely used tool in probability and statistics. For an extensive discussion, applications and generalizations the reader is referred to Tong (1980, Chapter 4). The purpose of this note is to show, by a counterexample, that the necessary and sufficient condition for equality, given by Anderson (1955, p. 172) and quoted in Tong (1980, p. 54), is incorrect and to give a valid necessary and sufficient condition and to apply it to the multivariate normal distribution.

2. A COUNTEREXAMPLE AND NECESSARY AND SUFFICIENT CONDITIONS FOR EQUALITY

We begin by establishing notation and stating assumptions. Let $\tilde{x} = (x_1, x_2, \ldots, x_k)$ be a point in R^k, $f(\tilde{x}) : R^k \to [0,\infty)$ be unimodal, symmetric about the origin and such that $\int_A f(\tilde{x})d\tilde{x} < \infty$, $A \subset R^k$ be convex, symmetric about the origin and have a nonempty interior, $D_u = \{\tilde{x} | f(\tilde{x}) \geq u\}$, $u > 0$, $A \cap D_u$ have a nonempty interior for at least one $u > 0$, $\tilde{y} \neq \tilde{0}$ an arbitrary vector in R^k, and $\lambda \in [0,1]$. Then Anderson's theorem (1955) states that

$$\int_{A+\lambda\tilde{y}} f(\tilde{x})d\tilde{x} \geq \int_{A+\tilde{y}} f(\tilde{x})d\tilde{x} \, , \tag{2.1}$$

with equality for all $\lambda \in [0,1)$ if and only if

$$[(A + \tilde{y}) \cap D_u] = [(A \cap D_u)] + \tilde{y} \tag{2.2}$$

for every $u > 0$.

We first consider a counterexample. Let $a > 0$, $k = 2$, $A = \{(x_1,x_2) | -a \leq x_1 \leq a\}$, $f(\tilde{x}) = (2\pi)^{-1} e^{-(x_1^2+x_2^2)/2}$, $\tilde{y} = (0,1)$. Here the

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D_u's are circles with center at the origin and clearly

$$[(A + \tilde{y}) \cap D_u] = [A \cap D_u] \neq [A \cap D_u] + \tilde{y} \text{ .} \tag{2.3}$$

However,

$$\int_A f(\tilde{x})d\tilde{x} = \int_{A+\lambda\tilde{y}} f(\tilde{x})d\tilde{x} \text{ ,} \tag{2.4}$$

$\lambda \in [0,1)$, contradicting Anderson's assertion that (2.2) is a necessary condition for (2.4). (2.4) also contradicts the assertion made by Tong (1980, p. 54) that if f is a multivariate normal density with positive definite covariance matrix Σ and mean vector $\tilde{0}$, then the inequality in (2.1) is always strict for $\lambda \in [0,1)$. While Tong attributes this statement to Anderson (1955), it does not seem to have been made by him.

We now state and prove a more general necessary and sufficient condition for equality in (2.1).

<u>Theorem 2.1</u>. A necessary and sufficient condition for equality in (2.1) (for all λ such that $\lambda \in [0,1)$) is that, for each $u > 0$, there exists a vector $\tilde{z} \in R^k$, which may depend upon u, such that

$$[(A + \tilde{y}) \cap D_u] = [A \cap D_u] + \tilde{z} \text{ .} \tag{2.5}$$

<u>Proof</u>. *From the proof of* Anderson's (1955) theorem, necessary and sufficient conditions for equality are (obtained by setting $\lambda = 0$ in (2.1))

$$[A \cap D_u] = \frac{1}{2} [(A + \tilde{y}) \cap D_u] + \frac{1}{2} [(A - \tilde{y}) \cap D_u] \text{ ,} \tag{2.6}$$

and

$$\frac{1}{2} [(A + \tilde{y}) \cap D_u] = \frac{1}{2} [(A - \tilde{y}) \cap D_u] + \tilde{z} \text{ ,} \tag{2.7}$$

where $\tilde{z}$ is an arbitrary vector which may depend on u. (2.7) is, for this case, the necessary and sufficient condition for equality in the Brunn-Minkowski inequality. It is readily verified that

$$[A \cap D_u] \supset \frac{1}{2} [(A + \tilde{y}) \cap D_u] + \frac{1}{2} [(A - \tilde{y}) \cap D_u] \tag{2.8}$$

and so only the reverse inclusion must be show in order for (2.6) to hold.

For sufficiency we must show that (2.5) implies (2.6) and (2.7). From the hypotheses,

$$-[(A + \tilde{y}) \cap D_u] = [(-A - \tilde{y}) \cap (-D_u)] = [(A - \tilde{y}) \cap D_u] \tag{2.9}$$

and from (2.5)

$$-[(A + \tilde{y}) \cap D_u] = -[(A \cap D_u) + \tilde{z}] = [(A \cap D_u)] - \tilde{z} \tag{2.10}$$

and so (2.9) and (2.10) imply

$$[(A - \tilde{y}) \cap D_u] = [(A \cap D_u)] - \tilde{z} \ . \tag{2.11}$$

From (2.5) and (2.11), the reverse inclusion in (2.6) follows and so (2.6) holds. (2.5) and (2.11) also immediately imply that (2.7) holds, thus proving sufficiency.

Consider now the necessity. We must show that (2.6) and (2.7) imply (2.5). From (2.7),

$$\frac{1}{2}\,[(A - \tilde{y}) \cap D_u] = \frac{1}{2}\,[(A + \tilde{y}) \cap D_u] - \tilde{z} \tag{2.12}$$

and substituting (2.12) in (2.6) gives

$$\begin{aligned}[A \cap D_u] &= \frac{1}{2}\,[(A + \tilde{y}) \cap D_u] + \frac{1}{2}\,[(A + \tilde{y}) \cap D_u] - \tilde{z} \\ &= [(A + \tilde{y}) \cap D_u] - \tilde{z} \ ,\end{aligned} \tag{2.13}$$

since $[(A + \tilde{y}) \cap D_u]$ is convex, and (2.13) is equivalent to (2.5), proving the necessity.

Note that the counterexample satisfies (2.5) with $\tilde{z} = \tilde{0}$. That $\tilde{z} = \tilde{0}$ in the counterexample is not an accident. We state a result to that end and indicate the proof.

<u>Corollary 2.1</u>. Let $[A \cap D_u] + \tilde{z} = [(A + \tilde{y}) \cap D_u]$. If $A \cap D_u = A$, then A is bounded and $\tilde{z} = \tilde{y}$. If $A \cap D_u = D_u$, then D_u is bounded and $\tilde{z} = \tilde{0}$. If $A + \tilde{y} = A$, then $A \cap D_u$ is bounded and $\tilde{z} = \tilde{0}$. If $k = 1$, $\tilde{z} = \tilde{0}$ or $\tilde{z} = \tilde{y}$.

Proof. If $A \cap D_u = A$, then A is bounded, since $\int_A f(\tilde{x})d\tilde{x} < \infty$ and A is convex. From the hypotheses,

$$[A \cap D_u] + \tilde{z} = A + \tilde{z} = [(A + \tilde{y}) \cap D_u] \tag{2.14}$$

and thus $A + \tilde{z} \subset A + \tilde{y}$ and $V(A + \tilde{z}) = V(A + \tilde{y})$ and therefore $A + \tilde{z} = A + \tilde{y}$ and so $\tilde{z} = \tilde{y}$, since A is bounded.

If $A \cap D_u = D_u$, then D_u is bounded, since $\int_A f(\tilde{x})d\tilde{x} < \infty$ and D_u is convex. From the hypotheses,

$$[A \cap D_u] + \tilde{z} = D_u + \tilde{z} = [(A + \tilde{y}) \cap D_u] \tag{2.15}$$

and thus $D_u + \tilde{z} \subset D_u$ and therefore $\tilde{z} = \tilde{0}$, since D_u is bounded.

If $A + \tilde{y} = A$, then $A \cap D_u$ is bounded, since $\int_A f(\tilde{x})d\tilde{x} < \infty$ and $A \cap D_u$ is convex. From the hypotheses,

$$[A \cap D_u] + \tilde{z} = [A \cap D_u] \tag{2.16}$$

and therefore $\tilde{z} = \tilde{0}$, since $A \quad D_u$ is bounded.

If $k = 1$, then either $A \cap D_u = A$ or $A \cap D_u = D_u$, giving the conclusion.

Note that for $k > 2$, $\tilde{z}$ can take values other than $\tilde{0}$ and $\tilde{y}$. For example, let $a > 0$, $k = 2$, $A = \{(x_1,x_2)|-2a < x_1 < 2a\}$,

$$f(x_1,x_2) = \begin{cases} a - |x_2|, & 0 < |x_2| < a \\ 0 & |x_2| > a \end{cases} \tag{2.17}$$

and $\tilde{y} = (y_1,y_2)$ be arbitrary. Then, for $u < a$,

$$D_u = \{(x_1,x_2) \mid |x_2| < a - u\} \tag{2.18}$$

and

$$[(A + \tilde{y}) \cap D_u] = [A \cap D_u] + (y_1,0) \; . \tag{2.19}$$

A similar result holds if A is bounded and f defined over a rectangle, provided $\tilde{y}$ is sufficiently small. Incidentally, the above is another counterexample to Anderson's necessary and sufficient condition.

Note also that there are other probability inequalities in Anderson (1955), based upon (2.1), and in them the conditions for equality must be changed in accordance with Theorem 2.1.

A possible difficulty with Anderson's (1955) proof of the necessity and sufficiency is that it seems to assume that when $[(A + y) \cap D_u]$ and $[(A - y) \cap D_u]$ are translates of each other, without loss of generality the translating vector may be taken as $2\tilde{y}$. The counterexample shows this is not necessarily true.

We now apply the above to the multivariate normal distribution.

<u>Theorem 2.2</u>. Let A be bounded and $f(\tilde{x})$ be the multivariate normal density with positive definite covariance matrix Σ and mean vector $\tilde{0}$. Then the inequality in (2.1) is strict for all $\lambda \in [0,1)$.

<u>Proof</u>. Without loss of generality we may assume that $\lambda = 0$ and $\Sigma = I$, since convexity is preserved under nonsingular linear transformations. Thus the D_u's are circles with center at the origin. Since A is bounded, there is a smallest u such that D_u contains A, say u_0. Then (2.5) is violated for u_0, giving the conclusion.

<u>Corollary 2.2</u>. If A is bounded and $f(\tilde{x})$ is elliptically contoured with nonincreasing g, i.e., $f(\tilde{x}) = |\Sigma|^{-1/2} g(\tilde{x}'\Sigma^{-1}\tilde{x})$, Σ positive definite, then the inequality in (2.1) is strict for all $\lambda \in [0,1)$.

<u>Proof</u>. g nonincreasing implies f is unimodal (Tong, 1980, p. 74) and then the proof is exactly as above.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2648	2. GOVT ACCESSION NO. AD-A141509	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A NOTE ON EQUALITY IN ANDERSON'S THEOREM		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Andrew P. Soms		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 4 - Statistics and Probability
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE February 1984
		13. NUMBER OF PAGES 6
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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